THEORY OF SHORT WAVES IN GAS DYNAMICS

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An examination is made of the two-dimensional, almost stationary flow of an ideal gas with small but clear variations in its parameters. Such gas motion is described by a system of two quasilinear equations of mixed type for the radial and tangential velocity components [1, 2]. Partial solutions [3, 4], characterizing the variation in the gas parameters in the vicinity of the shock wave front (in the short-wave region), are known for this system of equations. The motion of the initial discontinuity of the short waves derived from the velocity components with respect to polar angle and their damping are studied in the report. A solution of the equations characterizing the arrangement of the initial discontinuity derived from the velocities is presented for one particular case of the class of exact solutions of the two parameter type [4]. Functions are obtained which express the nature of the variation in velocity of the front of the damped wave and its curvature.

1. Formulation of the Problem. Let us write the equation of the short waves for two-dimensional, almost stationary streams of ideal gas [1] in the form

$$U_\delta + AU_y + B = 0$$
$$U = \begin{bmatrix} \mu \\ \nu \end{bmatrix}, \quad A = \begin{bmatrix} 0 & {}^1/_2(\mu - \delta) \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} k\mu/(\mu - \delta) \\ 0 \end{bmatrix} \tag{1.1}$$

Here k = 1 for motions having axial symmetry in the spherical coordinate system r, θ; k = 1/2 for plane parallel motions in the polar coordinate system r, θ; the dimensionless functions μ, ν, δ, and y are connected with the radial and tangential velocity components u and v and the independent variables by the equations

$$u = a_0 M_0 \mu, \; v = a_0 M_0 \, [{}^1/_2 (\gamma + 1) \, M_0]^{1/2} \nu$$
$$r = a_0 t \, [1 + {}^1/_2 (\gamma + 1) \, M_0 \delta], \; \theta = [{}^1/_2 (\gamma + 1) \, M_0]^{1/2} y$$

Let us examine the region in which the proper values $\lambda^{1,2} = \pm 1/\sqrt{2(\delta - \mu)}$ of the matrix A are real while the corresponding left-handed characteristic vectors $l^{1,2} = [1 \;\; \lambda^{1,2}]$ are linearly independent. Suppose the column vector U (δ, y) has a discontinuity in the first derivatives with respect to the variable y at some initial value $\delta = \delta_0$ when $y = y_0$ in the above-indicated region where system (1.1) is hyperbolic. The initial discontinuities of the derivatives move along the characteristic curves

$$dy \,/\, d\delta = \pm \, [2 \, (\delta - \mu)]^{-1/2}$$

of system (1.1). The law of motion of the initial weak discontinuity must be established, i.e., the nature of the dependence on the variable δ must be determined.

2. Equations for the Discontinuities. Jeffrey and Taniuti [5] studied in general form the problem of the distribution of the initial discontinuity in the derivatives for quasilinear hyperbolic systems.

Following [5], let us introduce the new independent variables

$$\varphi \, (\delta, \, y) = \text{const}, \quad \delta^1 = \text{const}$$

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setting

$$\delta^1 = \delta,\ \varphi_\delta + \lambda^1 \varphi_y = 0$$

Then the wave front is described by the equation $\varphi(\delta, y) = 0$, and the matrix equation (1.1) is represented in the form

$$l^{1,2}\{y_\varphi U_{\delta^1} + (\lambda^{1,2} - \lambda^1) U_\varphi + y_\varphi B\} = 0 \tag{2.1}$$

Let [E] designate the discontinuity $E_{\varphi=0^-} - E_{\varphi=0^+}$ in the quantity E. Then

$$[U] = 0,\ [U_{\delta^1}] = 0$$
$$[U_\varphi] = \Pi(\delta^1) \neq 0,\ [y_\varphi] = Y(\delta^1) \neq 0$$

Examining Eq. (2.1) to the right and left of the wave front, we obtain equations relating the unknown column vector

$$\Pi(\delta^1) = \begin{bmatrix} M \\ N \end{bmatrix} \text{ and the scalar } Y(\delta^1)$$

$$\begin{gathered} l_0^2 y_\varphi \Pi - l_0^2 U_{\varphi 0} Y = 0 \\ l_0^1 \Pi' + [(\nabla_U l^1)_0 \Pi]^* U_{0\delta^1} + [\nabla_U (l^1 B)]_0 \Pi = 0 \\ Y' = (\nabla_U \lambda^1)_0 \Pi \end{gathered} \tag{2.2}$$

Here a prime designates a derivative with respect to δ^1, an asterisk designates a transposition operation, the zero index designates the value in front of the wave front, and ∇_U designates the gradient operator in U space.

We substitute into Eq. (2.2) the proper values and proper vectors of the matrix A and the expressions for the column vectors U, Π, and B and obtain equations for the motion of the discontinuities M, N, and Y

$$M = [2(\delta - \mu)]^{3/2} Y' \tag{2.3}$$

$$N = [\nu_y + \sqrt{2(\delta - \mu)}\, \mu_y] Y - [2(\delta - \mu)]^2 Y' \tag{2.4}$$

$$\begin{gathered} 2[2(\delta - \mu)]^2 Y'' - \{\nu_y + \sqrt{2(\delta - \mu)}\, \mu_y - \\ - {}^7/_2 [2(\delta - \mu)][2(\delta - \mu)]' + \sqrt{2(\delta - \mu)}\, \nu_\delta + 4k\delta\} Y' - \\ - [\nu_y + \sqrt{2(\delta - \mu)} \times \mu_y]' Y = 0 \end{gathered} \tag{2.5}$$

The null indices and the unit on the variable δ are dropped from Eqs. (2.3)-(2.5).

3. Solution of Equations for the Discontinuities. For application it is important to solve the short-wave equations in symmetrical form. Let us use one case of the two-parameter class of exact solutions [4] of the system (1.1)

$$\mu = \xi,\ \nu = c_1 y,\ \delta = \chi(\xi) \tag{3.1}$$

$$\chi(\xi) = \begin{cases} c(\xi + c_1)^2 + 2(\xi + c_1) - c_1 & \text{at } k = {}^1/_2 \\ -c_1/2 - (\xi + c_1/2) \ln c(\xi + c_1/2) & \text{at } k = 1 \end{cases}$$

In this case Eq. (2.5) reduces to a first-order equation for the function Z = Y'

$$\ln Z c_2^{-1} + \frac{7 - 2k}{4} \ln [2(\delta - \xi)] + \frac{c_1 + 4k\xi}{8(\delta - \xi)} = \int \left[\frac{c_1 + 4k\xi}{8(\delta - \xi)^2} + \frac{k}{\delta - \xi}\right] d\xi \tag{3.2}$$

with the solution

$$Z = c_2 [1 + C\eta]^{-(9 + cc_1)/4}\ \eta^{(cc_1 - 3)/4} \exp(c_1 / 4\eta) \tag{3.3}$$

at k = 1/2 and

$$Z = [c c_2 2^{5/4}]^{-1} [\delta - \xi]^{-9/4}\, \eta \exp\{-cc_1 / 8\eta\ \Psi(1, 1, 1 + \ln \eta)\} \tag{3.4}$$

at k = 1.

Here and later $\Psi(a, b, x)$ is a degenerate hypergeometric function related to the incomplete gamma function [6]

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1}\, dt$$

$$\Gamma(\alpha, x) = x^\alpha e^{-x} \Psi(1, 1+\alpha, x) = e^{-x} \Psi(1-\alpha, 1-\alpha, x)$$

where c_2 is a constant of integration

$$\eta = \begin{cases} \xi + c_1 & \text{at } k = 1/2 \\ c(\xi + c_1/2) & \text{at } k = 1 \end{cases}$$

The solutions of Eqs. (3.2) and (3.3) and of (3.2) and (3.4)

$$Y = 2c_2 \left[\frac{\delta - \xi}{\eta^2}\right]^{-(c_2c_1+1)/4} e^{c_1/4\eta} \Psi\left(1, \frac{3 - cc_1}{4}, \frac{c_1(\xi - \delta)}{4\eta}\right) + c_3$$

$$Y = [c_2 2^{5/4}]^{-1} \int (\delta - \xi)^{-5/4} \exp\left\{-\frac{cc_2}{8\eta} \Psi(1, 1, 1 + \ln\eta)\right\} d\xi + c_3$$

represent the laws of motion of the discontinuity Y for k=1/2 and 1.

In the latter case for uniform gas motions $c_1 = 0$, and the solution takes the form

$$Y = [c_2 2^{5/4} \xi (\delta - \xi)^{1/4}]^{-1} \{(\delta - \xi) \Psi(1, 7/4, (\delta - \xi)/4\xi) - 4\xi\} + c_3$$

The laws of motion of the discontinuities in the derivatives with respect to velocity are obtained from (2.3) and (2.4):

$$M = [2(\delta - \xi)]^{3/2} Z, \quad N = c_1 Y - [2(\delta - \xi)]^2 Z$$

into which it is necessary to insert the corresponding values of δ, Y, and Z for k=1/2 and 1.

4. Damping of Short Waves. Let us change to the variables δ^1 and τ in the equations (1.1) according to the equations

$$\delta^1 = \delta, \quad \tau = y - \sqrt{2\delta}$$

$$\frac{\partial}{\partial\delta} = \frac{\partial}{\partial\delta^1} - \frac{1}{\sqrt{2\delta}} \frac{\partial}{\partial\tau}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial\tau}$$

and let us represent the unknown velocities μ and ν in the vicinity of $\tau = 0$ in the form of expansions by powers of τ:

$$\begin{aligned} \mu &= \tau\mu_1(\delta) + \tau^2\mu_2(\delta) + \ldots \\ \nu &= \nu_0 + \tau\nu_1(\delta) + \tau^2\nu_2(\delta) + \ldots \end{aligned} \tag{4.1}$$

Having substituted (4.1) into the transformed equations and equated the coefficients for the same powers of τ, we obtain a first-approximation equation

$$\nu_1 + \sqrt{2\delta}\, \mu_1 = 0, \tag{4.2}$$

a second-approximation equation

$$2\mu_2 + \frac{2}{\sqrt{2\delta}} \nu_2 - \nu_1' = 0 \tag{4.3}$$

$$\frac{1}{\sqrt{2\delta}} = \frac{1}{\nu_1'}\left[\mu_1' - \frac{\mu_1\nu_1}{2\delta^2} - \frac{k}{\delta}\mu_1\right], \tag{4.4}$$

and a third-approximation equation

$$\begin{gathered} 3\mu_3 + 3/\sqrt{2\delta}\, \nu_3 - \nu_2' = 0 \\ \frac{1}{\sqrt{2\delta}} = \frac{1}{\nu_2'}\left\{\mu_2' - \frac{1}{2\delta}\left[\frac{2\mu_1\nu_2}{\delta} + \nu_1\left(\frac{\mu_2}{\delta} + \frac{\mu^2_1}{\delta^2}\right)\right] - \frac{k}{\delta}\left(\mu_2 + \frac{\mu_1^2}{\delta}\right)\right\} \end{gathered} \tag{4.5}$$

The solution of Eqs. (4.2) and (4.4)

$$\mu_1 = \delta^{1/2}\left[\frac{\sqrt{2}}{2k - 3} - c_1\delta^{(3-2k)/4}\right]^{-1}$$

$$\nu_1 = -\sqrt{2\delta}\, \mu_1$$

characterizes the damping of the rate of change in the wave front. The solution of Eqs. (4.3) and (4.5)

$$\mu_2 = \frac{A}{(4\sqrt{2}\,\omega)^3} t^{(2-k)/\omega} \left(\frac{\nu_1}{\delta}\right)^3$$

$$\nu_2 = {}^1/_2 \sqrt{2\delta}\,(\nu_1' - 2\mu_2)$$

characterizes the damping of the curvature of the wave front.

Here

$$A = \omega \left[\frac{a}{k-2+2\omega} t^2 + \frac{b}{k-2+\omega} t + \frac{c_2}{k-2}\right] t^{(k-2)/\omega} + c_3$$

$$t = \delta^\omega, \quad \omega = (3-2k)/4$$

$$a = (4\omega c_1)^2 (1 - 3\omega + 2\omega^2)\, 2^{-1/2}$$

$$b = 4\omega c_1 [2 + (8k - 11)\omega + 6\omega^2]$$

$$c_2 = \sqrt{2}\,[1 - 8(1-k)\omega + 16\omega^2]$$

and c_1 and c_3 are constants of integration.

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